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Machinery Research and Development Directorate
Technical Report

Some Implications of a Differential Turbomachinery Equation with Viscous Correction

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ABSTRACT

A differential equation describing the energy transfer between a fluid and a body moving in that fluid was derived. The derivation, based upon the Coriolis form of the Navier-Stokes Equation, contains a rigorous viscous correction. For inviscid ideal cases, the equation demonstrates that the rate of total enthalpy transfer from (or to) the system is a function of the transverse component of the pressure gradient. Therefore, for practical turbomachinery rotors, the derivative, $\partial p/\partial \theta$, can never vanish.

On integration of the differential equations, a form of the Euler Turbomachinery Equation with viscous correction is derived. The resultant form contains two distinct work rate terms for the axial and radial components of the flow. The fact that integration yields a result which approximates the classic Euler Turbomachinery Equation constitutes confirmation of the derivation.

An application of the equation to an ideal infinite linear cylinder with bound vorticity was developed, yielding the expected known result.

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FIGURES

ADMINISTRATIVE INFORMATION

This report has been accepted for publication by the Journal of the American Institute of Aeronautics and Astronautics, provided that it be synopsized. Its publication will provide useful source material for readers who wish to examine the original unexpurgated scientific arguments. The report is a further development of earlier work (1987) performed in partial fulfillment of thesis requirements of the Aerospace Department of the University of Maryland at College Park. Professor Everett Jones was the graduate advisor and director. Publication of this manuscript was supported by the Naval Surface Warfare Center, Carderock Division, Annapolis Detachment under the Independent Research and Development Program (IR&IED) of Dr. Bruce Douglas.

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NOMENCLATURE

f	any continuous function
f_r	radial fraction of the mass flow rate
f_z	axial fraction of the mass flow rate
h	enthalpy per unit mass of fluid
h_o	total enthalpy per unit mass of fluid
H_o	total enthalpy of a fluid
H_o'	total enthalpy of a fluid per unit length of blade
I	relative total specific rothalpy
L	lift per unit length of blade
m	mass flow rate
m_r	radial mass flow rate
m_z	axial mass flow rate
p	local static pressure
p_o	total pressure
q	a generalized curvilinear coordinate
$\dot{q}$	a specific heat rate
r	radial coordinate of a cylindrical coordinate system
s	entropy per unit mass of fluid
t	time
T	absolute temperature
$\bar{U}$	velocity of the blade and a function of the radius
$\bar{V}$	velocity
(VIS)	the integrated viscous term of Equation (22)
$\bar{W}$	relative fluid velocity in a moving rotor frame
$\bar{W}'$	a time-dependent component of the velocity in the moving frame
z	axial coordinate of a cylindrical coordinate system
Γ	the circulation
θ	tangential angle coordinate in cylindrical coordinates
ν	the kinematic viscosity
$\bar{\bar{\pi}}'$	the stress tensor separated from the thermodynamic pressure
ϱ	local fluid density
τ	volume
ψ	a stream function
$\bar{\varpi}$	angular velocity vector

INTRODUCTION

A monument of turbomachinery technology, the Turbomachinery Equation of Euler is based upon thermodynamic definitions of work and Newton's Laws. Since the Navier-Stokes Equations and Crocco's Equation [1,2] in a rotating (moving) frame are also based upon thermodynamics and Newton's Laws, they must in principle contain the Turbomachinery Equation in differential form and, on integration, in integral form. Application of Coriolis' transformation to the viscous form of Crocco's Equation leads to an uncoupled expression for the substantial derivative of the total enthalpy in a differential turbomachinery equation which, on integration, yields a novel form of the Turbomachinery Equation corrected for viscous non-ideal flow.

The classical Turbomachinery Equation is an integrated expression arising from Euler's analysis of fluid torque and mechanical shaft work transferred between a fluid and a rotor moving in that fluid. Since heat transfer is negligible in most turbomachines, the shaft work has been equated to the specific total enthalpy transfer, Euler's Turbomachinery Equation (see Horlock [3], page 77, Equation 4.3) as follows:

$$\Delta h_o = \Delta(UV_U) \ , \qquad (1)$$

where V_U is the component of the absolute velocity, $\bar{V}$, in the direction $\bar{U}$ of the rotor (or energy-transferring device).

Whether one interprets the quantity in the right member of (1) as shaft work or as a change in the specific total enthalpy, the right member is a function only of the end points of the integration. Therefore, the integral is an exact or total integral, and it represents a function of state. However, in non-conservative, dissipating systems, attempts to express the thermodynamic process in terms of the end points alone leads to thermodynamic inconsistencies. An example of such inconsistencies arises in the consideration of a propeller windmilling on a frictionless shaft in a moving fluid. Since the shaft delivers no work, a viscous correction is necessary whether the right member of (1) represents work or a change in the specific total enthalpy.

These points are discussed in more detail in terms of the new differential equation. Also, an application of the differential turbomachinery equation is described for a two-dimensional, ideal, linear turbine.

THE TRANSFORMATION BETWEEN ABSOLUTE AND MOVING FRAMES

In the following discussion the subscripts v and w represent the absolute and the moving frame coordinate and vector values. (See Figure 1 and the Nomenclature for definitions of quantities.) A frequently used relation connecting the absolute and moving frame velocities is

$$\bar{V} = \bar{W} + \bar{U} \ . \qquad (2)$$

Following Spannhake [4], spatial derivatives in the moving frame and time derivatives have the following relationship:

The Transformation

$$t_v = t_w = t$$

$$z_v = z_w + z_0 = z_w = z \text{ if } z_0 = 0$$

$$r_v = r_w = r$$

$$\theta_v = \theta_w + \omega t$$

$$\overline{R}_v = \overline{R}_w + r\omega t\bar{e}_\theta$$

$$\overline{V} = \overline{W} + \overline{U}$$

$$\left(\frac{\partial}{\partial\theta_w}\right)_{\theta_v} = -\frac{1}{\omega}\left(\frac{\partial}{\partial t}\right)_{\theta_v}$$

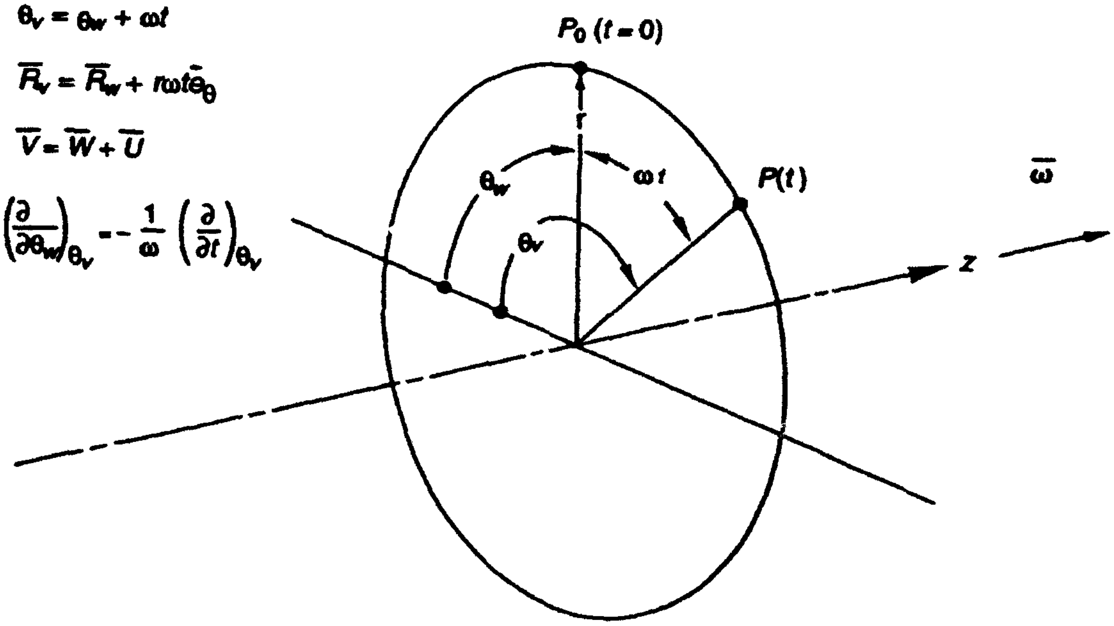

Figure 1. Configurational relationships between the absolute and moving coordinate systems.

$$\frac{1}{r}\left(\frac{\partial}{\partial\theta_w}\right)_{\theta_v} = -\frac{1}{r\omega}\left(\frac{\partial}{\partial t}\right)_{\theta_v} = -\frac{1}{U}\left(\frac{\partial}{\partial t}\right)_{\theta_v} . \tag{3}$$

Similar results are obtained for non-rotating systems with cartesian coordinates.

$$\left(\frac{\partial}{\partial y_w}\right)_{y_v} = -\frac{1}{U}\left(\frac{\partial}{\partial t}\right)_{y_v} . \tag{4}$$

Equations (3) and (4) define, in fact, the crypto-steady criterion [5], which, if U is constant, indicates that a frame exists in which the flow regime may be truly steady state.

Since the vector operator ∇ is independent of time and since vector operators are independent of frame,

$$\nabla_v = \nabla_w = \nabla . \tag{5}$$

Now from Equation (3) for any static function f,

$$\left(\frac{\partial f}{\partial t}\right)_{q_i} = -U\left(\frac{\partial f}{\partial q_{wu}}\right) = -\bar{U}\cdot\nabla_w f = -\bar{U}\cdot\nabla f . \tag{6}$$

where q_i represents all the position coordinates in the absolute frame and q_{wu} is the moving-frame coordinate in the direction of motion of the lifting body (such as the θ direction for a rotor). Equation (6) is an extension of the usual crypto-steady relation.

To appreciate the significance of the final equation of this section, consider an observer located on a blade of a rotating windmill in an infinite fluid which at infinity translates with uniform constant velocity. The observer cannot detect any time-dependent changes in the fluid at any given point on the blade. However, on moving to the absolute frame, the observer notes changes in velocity, pressure and temperature as each blade passes (see Dean [6]). Thus, the inequality

$$\left(\frac{\partial}{\partial t}\right)_{\theta_v} \neq \left(\frac{\partial}{\partial t}\right)_{\theta_w} \tag{7}$$

indicates that in the moving frame all partial time derivatives may vanish while the absolute partial time derivative may be finite.

Note, however, in contrast with the inequality (7) that the substantial derivative of any static quantity at a point is independent of the coordinate system or frame and is therefore an invariant with respect to frame.

DERIVATION OF THE DIFFERENTIAL AND INTEGRAL FORMS

UNCOUPLING THE SUBSTANTIAL TOTAL ENTHALPY DERIVATIVE IN THE MOVING AND ABSOLUTE FRAMES

Relationships between absolute and relative flow fields are given by the Coriolis form of the Navier-Stokes Equations [3,7–11].

$$\frac{\partial \overline{V}}{\partial t} + \nabla V^2/2 - \overline{V} \times (\nabla \times \overline{V})$$
$$= \frac{\partial \overline{W}}{\partial t} + \nabla W^2/2 - \overline{W} \times (\nabla \times \overline{W}) + 2\overline{\omega} \times \overline{W} - \nabla U^2/2$$
$$= \frac{-\nabla p}{\varrho} + \frac{1}{\varrho} \nabla \cdot \overline{\overline{\pi}}' , \qquad (8)$$

where $\overline{\overline{\pi}}'$ represents the stress tensor excluding the pressure tensor $p\delta_i^j$, and $\overline{U}$ is independent of time. The relative acceleration, $\partial \overline{W}/\partial t$, is defined in the relative frame; i.e., $(\partial \overline{W}/\partial t)_w$.

Using the absolute-frame equality of (8) and the gradient form of Gibbs' equation of state, the substantial derivative of the total enthalpy is obtained in terms of the partial derivative of the pressure, the substantial derivative of the entropy and the stress tensor $\overline{\overline{\pi}}'$. See Wu et al. [7–9].

$$\frac{Dh_o}{Dt} = \frac{1}{\varrho}\left(\frac{\partial p}{\partial t}\right)_v + T\frac{Ds}{Dt} + \frac{\overline{V}}{\varrho} \cdot (\nabla \cdot \overline{\overline{\pi}}') . \qquad (9)$$

At this point non-ideal analysis stops unless the partial derivative of the pressure with respect to time is resolved in terms of the pressure gradient [10]. Employing Equation (6) with the static pressure as the arbitrary function the result is

$$\frac{Dh_o}{Dt} = -\frac{\overline{U}}{\varrho} \cdot \nabla p + T\frac{Ds}{Dt} + \frac{\overline{V}}{\varrho} \cdot (\nabla \cdot \overline{\overline{\pi}}') = -\frac{\omega}{\varrho}\frac{\partial p}{\partial \theta} + T\frac{Ds}{Dt} + \frac{\overline{V}}{\varrho} \cdot (\nabla \cdot \overline{\overline{\pi}}') . \qquad (10)$$

Equation (10) may be simplified further using Wu's (reference [8] page 91) definition of the substantial derivative of the entropy in the moving frame following a particle of fluid (the entropy and its substantial derivative are invariant with respect to frame). Thus:

$$T\frac{Ds}{Dt} = \dot{q} + \frac{1}{\varrho}\overline{\overline{\pi}}' : \nabla \overline{W} , \qquad (11)$$

where $\dot{q}$ represents heat transfer, and the second term is the specific dissipation. Then from (10) and (11), eliminating the substantial derivative of the entropy, one obtains [10]

$$\frac{Dh_o}{Dt} = -\frac{\omega}{\varrho}\frac{\partial p}{\partial \theta} + \frac{1}{\varrho}\nabla \cdot (\bar{\bar{\pi}}' \cdot \bar{W}) + \frac{1}{\varrho}\bar{U} \cdot (\nabla \cdot \bar{\bar{\pi}}') + \dot{q} \quad . \tag{12}$$

Now Equation (8) may be used to replace the pressure gradient in (12). From the dot product of $\bar{U}$ with the moving frame equality in (8) the pressure gradient term becomes (see Reference [10]):

$$-\bar{U} \cdot \frac{\nabla p}{\varrho} = -\frac{\omega}{\varrho}\frac{\partial p}{\partial \theta} = \bar{U} \cdot \frac{\partial \bar{W}}{\partial t} + \bar{U} \cdot (\bar{W} \cdot \nabla)\bar{W} + 2\bar{U} \cdot \bar{\varpi} \times \bar{W}$$

$$-\bar{U} \cdot \nabla U^2/2 - \frac{1}{\varrho}\bar{U} \cdot (\nabla \cdot \bar{\bar{\pi}}') \quad . \tag{13}$$

The gradient of the static pressure is of course invariant in all frames, but the moving frame equation in (13) is convenient for later integration. Combining (12) with (13), the substantial derivative of the total enthalpy uncoupled from the entropy and pressure is obtained.

$$\frac{Dh_o}{Dt} = \bar{U} \cdot \frac{\partial \bar{W}}{\partial t} + \bar{U} \cdot (\bar{W} \cdot \nabla)\bar{W} + 2\bar{U} \cdot \bar{\varpi} \times \bar{W} + \frac{1}{\varrho}\nabla \cdot (\bar{\bar{\pi}}' \cdot \bar{W}) + \dot{q} \quad . \tag{14}$$

Equation (14), the essential development of this paper, is a differential turbomachinery equation expressed in terms of the moving frame velocity vector and $\dot{q}$. It is universally applicable in any moving frame whether rotating or not.

Referring to Equation (12), a particularly simple statement is obtained for ideal inviscid flow:

$$Dh_o/Dt = -\bar{U} \cdot \nabla p/\varrho = -(\omega \partial p/\partial \theta)/\varrho \tag{15}$$

The second equality in (15) is specific to three-dimensional rotating machines. Contrary to Reference [3], pp. 7–8, only the transverse pressure gradient term survives in an inviscid system, indicating that neither axial nor radial pressure gradients are germane to the calculation of specific total enthalpy transfer. (Mass flow rates are of course a function of axial or radial pressure gradients.) The impulse stages of turbomachines prove that axial or radial pressure gradients need play no role in energy transfer.

The conclusion that there can be no energy transfer without a transverse pressure gradient is implicit in the logic of Euler, who insisted that only the transverse component of the momentum is effective. Nevertheless, the assumption that transverse pressure gradients vanish*is popular in streamline curvature methods of design because it permits a simplified two-dimensional calculation of blade design [7–9, 11].

Equation (14) permits a revisiting of Dean's unsteadiness paradox [6] if one enquires into the possibility of rotor energy transfer between bladeless discs or concentric

*It is argued that for an infinite number of blades, dp becomes an infinitesimal, and therefore $\partial p/\partial \theta$ vanishes which, of course, violates the rules of calculus.

cylinders $(U = 0)$. (Dean's paradox is resolved most simply by reference to Equation (6)). A steady state prevails in both the absolute and the moving frame, and $(\partial W/\partial t)$ vanishes. However, the viscous term in (14) does not vanish, thus providing a mechanism for energy transfer.

$$\frac{Dh_o}{Dt} = -\frac{1}{\varrho}\bar{\bar{\pi}}' : \overline{W} + \frac{1}{\varrho}\overline{W} \cdot (\nabla \cdot \bar{\bar{\pi}}') + \dot{q} \quad . \tag{16}$$

Mechanical energy transfer is accomplished by the second viscous work term of the right member in (16). It is necessary in steady-state systems that the viscous dissipation and the heat loss balance.

The ideal mechanisms of energy transfer contained within $\overline{U} \cdot \nabla p/\varrho$ and defined by (13) through (15) in their ideal limit, include a term based upon the Coriolis force among others. Some writers (see pp. 115–117 of Reference [11]) have suggested that the Coriolis term is the necessary and sufficient term for energy transfer. If this were indeed true, then axial machines and linear quasi two-dimensional machines with vanishing ω, such as the wings of planes and the sails of ships would be useless.

THE TOTAL ROTHALPY ARGUMENT

Wu's total relative rothalpy relation [7–9] (see also [12]) with I defined by $h + W^2/2 - U^2/2 = h_o - UV_U$ is given by:

$$\left(\frac{DI}{Dt}\right)_w = \frac{1}{\varrho}\left(\frac{\partial p}{\partial t}\right)_w + \frac{1}{\varrho}\nabla \cdot (\overline{W} \cdot \bar{\bar{\pi}}') + \dot{q} \quad . \tag{17}$$

Consider a system in which the observer is fixed to a blade of an isolated rotating air screw or a marine screw in an infinite uniform fluid. No time-dependence can be sensed in the moving frame and a crypto-steady-state prevails [6]. The moving-frame partial derivative of the pressure with time must vanish. This poses a great simplification which justifies common practice [13] in the design of marine screws where time dependence (and heat transfer) are generally ignored.

Now, if viscosity, heat transfer and the time-dependent terms vanish, (17) reduces to

$$\overline{W} \cdot \nabla I = 0 \tag{18}$$

The well known non-trivial solution is

$$\nabla I = \nabla\left(h + W^2/2 - U^2/2\right) = \nabla(h_o - UV_U) = 0 \ . \tag{19}$$

Equations (19) represent gradient forms of the classic Turbomachinery Equation in the moving and absolute frame. Integration of the right member of (19) over a stream tube in the absolute frame yields the classic Turbomachinery Equation (1). The argument demonstrates that Equation (1) is strictly true only for isentropic and steady-state flow in a stream tube. However, in systems with dissipation, it is necessary to account for non-ideal effects in (1).

MOVING FRAME TIME DEPENDENCE

Consider in Equation (14) a small sinusoidal moving-frame oscillation imposed on the relative velocity $\overline{W}$ leading to a new velocity, $\overline{W} + \overline{W}'$. The ratio of the magnitude of time-dependent $\overline{W}'$ to time-independent $\overline{W}$ will be arbitrarily fixed at ≤ 0.05 . The imposed frequency of the time-dependent $\overline{W}'$ is at least an order of magnitude greater than the blade velocity.

With Reynolds averaging, the partial derivative of $\overline{W}'$ with respect to time will vanish. However, the Reynolds average over non-linear terms will lead to non-vanishing Reynolds stress terms which will not be addressed here. As a result of the Reynolds time-averaging process, Equation (14) may be written

$$\left\{\frac{Dh_o}{Dt}\right\} = \overline{U} \cdot \left\{(\overline{W} \cdot \nabla)\ \overline{W}\right\} + 2\overline{U} \cdot \overline{\omega} \times \left\{\overline{W}\right\} + \left\{1/\varrho\ \nabla \cdot \bar{\bar{\pi}}' \cdot \overline{W}\right\} + \left\{\ \dot{q}\ \right\}, \qquad (20)$$

where the braces represent the Reynolds time average and $\overline{W}$ and $\bar{\bar{\pi}}'$ include the time-dependent $\overline{W}'$. This notation will not be used again.

In the Reynolds time-averaging process, the acceleration term drops out as expected. Elimination of the acceleration term simplifies the integration of (14). As mentioned, heat transfer is generally not significant in most rotors [11, 13], and in particular, those associated with pumps and marine screws, and it will be ignored here.

Now, note that arbitrary samples of turbulent flow over short (relative to the blade) periods may be written as a Fourier summation of regular sinusoidal oscillations. For the limited picture developed here, Equation (20) is again applicable according to the above argument.

INTEGRATION OF THE TOTAL ENTHALPY RATE

A proper test of (14) would be whether, on integration over the rotor blade-to-blade flow, it would predict a total enthalpy transfer compatible with that of the classic Turbomachinery Equation (1). Therefore, the integration of (14) for crypto-steady conditions will be performed as a test in the three-dimensional domain. Then an application of the new equation will be developed in a two-dimensional linear turbine.

In the integration process it will be assumed that the flow may be divided into streams which pass between a given pair of blades. In the rotating frame the streamtube walls are fixed steady-state walls associated with a steady-state mass flow rate m which may consist of radial and axial mass flow components.

DERIVATION OF THE INTEGRAL FORM FROM THE DIFFERENTIAL FORM

The substantial derivative of the total enthalpy represents power transfer at any point in a flow field between the fluid and a body moving at constant speed in that fluid. To obtain the total enthalpy transfer rate over the entire field, it is necessary to integrate (14) thus:

$$\iiint \varrho \frac{Dh_o}{Dt} d\tau$$

$$= \iiint \varrho \left[\overline{U} \cdot (\overline{W} \cdot \nabla)\overline{W} + 2\overline{U} \cdot \overline{\omega} \times \overline{W}\right] rdrd\theta dz$$

$$+ \iiint \nabla \cdot (\overline{\overline{\pi}} \cdot \overline{W})\ rdrd\theta dz \ , \tag{21}$$

where the second term of the right member is the viscous term. The first term of the right member of (21) is the tangential component of the convection term obtained on dot multiplication with $\overline{U}$, i.e.,:

$$\varrho \overline{U} \cdot (\overline{W} \cdot \nabla)\overline{W}$$

$$= \varrho U\left(W_r \frac{\partial W_\theta}{\partial r} + \frac{W_\theta}{r}\ \frac{\partial W_\theta}{\partial \theta} + W_z \frac{\partial W_\theta}{\partial z} + \frac{W_r W_\theta}{r}\right) . \tag{22}$$

The first and fourth terms of (22) will be combined in an integral identified by $I_{1,4}$ thus:

$$I_{1,4} = \iiint \varrho r\omega \frac{W_r}{r}\ \frac{\partial r W_\theta}{\partial r} rdrd\theta dz$$

$$= \int\left(\iint \varrho W_r r d\theta dz\right) \frac{\partial r\omega \bar{\bar{W}}_\theta}{\partial r} dr \ , \tag{23}$$

where we have used the mean value theorem to take $\bar{\bar{W}}_\theta\ (r)$ outside the double integral. Thus:

$$\bar{\bar{W}}_\theta(r) = \frac{1}{\Delta\theta\Delta z} \int \int_{\theta_{min}}^{\theta_{max}} W_\theta\ d\theta dz \ . \tag{24}$$

The factor in parentheses in the right member of (23) is the radial mass flow at any point r:

$$m_r(r) = \iint \varrho W_r r d\theta dz = m f_r(r) \ , \tag{25}$$

where $f_r(r)$ is the radial fraction of the mass flow rate m. Then

$$I_{1,4} = m \int f_r(r) \frac{\partial r\omega \bar{\bar{W}}_\theta}{\partial r}\, dr = m \int f_r d(U\bar{\bar{W}}_\theta). \tag{26}$$

Now, in the steady-state the mean value of $f_r(r)$, $\bar{f}_r$, is a constant given by

$$\bar{f}_r = \int_{\text{entrance}}^{\text{exit}} f_r(U\bar{\bar{W}}_\theta) d(U\bar{\bar{W}}_\theta)/\Delta(U\bar{\bar{W}}_\theta) \ , \tag{27}$$

and,

$$I_{1,4} = m\bar{f}_r\Delta_r(U\bar{\bar{W}}_\theta) = \bar{m}_r[(U\bar{\bar{W}}_\theta)_{r_2} - (U\bar{\bar{W}}_\theta)_{r_1}] \tag{28}$$

where $\bar{m}_r$ is the mean value of m_r. Using the same arguments as used for the radial mass flow, the integral of the third term of the right member of (22) may be written to show the axial mass flow rate $m_z(z)$ explicitly. With the velocities, $\bar{\bar{W}}_z$, $\bar{\bar{W}}_\theta$, and $\bar{U}$ averaged over r and θ, one obtains the steady-state mean value, $\bar{m}_z$, and

$$I_3 = \bar{m}_z[(\bar{U}\bar{\bar{W}}_\theta)_{z_2} - (\bar{U}\bar{\bar{W}}_\theta)_{z_1}] \ . \tag{29}$$

The second term of the right member of (22) provides an integral, I_2, which contains the tangential kinetic energy.

$$I_2 = \int\int \frac{\varrho\omega r}{2}\left(\int \frac{\partial W_\theta^2}{\partial\theta} d\theta\right) drdz$$

$$= \int\int \frac{\varrho\omega r}{2}\left[W_\theta^2(\theta_2) - W_\theta^2(\theta_1)\right] drdz = 0 \ . \tag{30}$$

Since the tangential velocities at the blade walls are the blade velocity, the integral vanishes.

Now identifying the second term of the first integral in the right member of (21 as I_5, we may write using the above arguments

$$I_5 = \int\int\int 2\varrho\omega^2 r W_r r dr d\theta dz = \bar{m}_r[(U^2)_{r_2} - (U^2)_{r_1}] \ . \tag{31}$$

Summing the components of integration (28) through (31)

$$\iiint \varrho \frac{Dh_o}{Dt} d\tau = \bar{m}_r\left[\Delta_r(U\bar{\bar{W}}_\theta) + \Delta_r(U^2)\right]$$

$$+ \bar{m}_z\Delta_z(\bar{U}\bar{\bar{W}}_\theta) + (VIS) \;, \tag{32}$$

where Δ_r and Δ_z represent the change along r and z respectively, and (VIS) is the integrated viscous term.

Now adding $\bar{m}_z\Delta_z\bar{U}^2$ which is zero to (32),

$$\iiint \varrho \frac{Dh_o}{Dt} d\tau = \bar{m}_r\Delta_r\left[(U + \bar{\bar{W}}_\theta)U\right]$$

$$+ \bar{m}_z\Delta_z\left[(\bar{U} + \bar{\bar{W}}_\theta)\bar{U}\right] + (VIS) = m\Delta h_o \;. \tag{33}$$

In (33) the terms $\bar{\bar{W}}_\theta$ are averaged over θ and z in the first term and over r and θ in the second term. Finally,

$$\Delta h_o = \bar{f}_r\Delta_r(\bar{U}\bar{\bar{V}}_\theta) + \bar{f}_z\Delta_z(U\bar{\bar{V}}_\theta) + (VIS)/m = \Delta(\bar{U}\bar{\bar{V}}_\theta) + (VIS)/m \;. \tag{34}$$

In the right member of (34) the term (VIS) contains the viscous work as well as the dissipation, which may provide a net outflow of energy in viscously-coupled bladeless devices. The delta term which approximates the right member of Euler's Equation (1) may be considered a pseudo-ideal term.

Equation (34) exhibits some similarity with the ideal Equation (1). The fact that mathematical spatial averaging processes have been employed suggests that the pseudo-ideal term cannot be a function of the end points alone. Therefore, the pseudo-ideal term cannot be ideal unless the system is ideal. Similar conclusions apply to the total relative rothalpy [12] , which depends upon the pseudo-ideal term.

It is easy to combine the axial and radial flow terms in (34) by letting the operator Δ vary in both r and z . The derivation supports the conclusion that Equation (14) is indeed a differential turbomachinery equation.

A two-dimensional application and test of the differential form (14) on an ideal linear device where the solution is known precisely will now be examined.

THE SUBSTANTIAL TOTAL ENTHALPY RATE IN A TWO-DIMENSIONAL DEVICE

An infinite circular cylinder with bound circulation, as shown in Figure 2, is an elemental linear turbine. It may be considered as an infinite sail on a sailboat or an infinite wing on a sailplane. The device extracts energy from the ideal inviscid working fluid. Work is performed on the sailplane (fixed to a vertical rail) by raising its height at uniform speed U against gravity. Work on the sailboat is performed by moving the boat at uniform speed U which elevates a weight attached at minus infinity by an infinite tether.

In the moving frame the apparent velocity of the ideal working fluid at infinite distance is W_o. The relationship between the absolute and moving coordinate system and the velocities is given by the transformation of Figure 1.

Preston [14] computed energy transfer in two-dimensional arrays of ideal vortex points by transforming their potential functions to the absolute frame where they exhibit time dependence. The study involves circular cylinders of finite dimensions with bound vorticity which can be transformed into two-dimensional airfoils. Energy transfer is developed in steady-state moving frames by integrating Equation (14) over all space.

THE TOTAL ENTHALPY TRANSFER RATE BASED ON AERODYNAMICS

Since the flow field is ideal, the flow domain may be described by a potential function or its conjugate stream function. The lift is therefore the ideal lifting force, L, of the Kutta-Joukowski Equation given by

$$L = \varrho W_o \Gamma \ , \tag{35}$$

where Γ is the scalar circulation. The units are force per unit length of cylinder. In the absolute and moving frame the lift component L_y directed parallel to the y axis of Figure 2 is given by

$$L_y = \varrho W_{ox} \Gamma = \varrho V_{ox} \Gamma \ , \tag{36}$$

where the subscript x represents the x component. Recalling that U is the velocity of motion of the device (sail or wing or rotating cylinder) as perceived in the absolute frame, the power is the product of U and L_y.

$$Power/unit\ length = \varrho U W_{ox} \Gamma \ . \tag{37}$$

Since we assume that there is no heat rate,

$$\frac{DH_o'}{Dt} = -\varrho U W_{ox} \Gamma \ , \tag{38}$$

where H_o' is the total enthalpy of the system per unit length of lifting surface. Equation (38) is the anticipated relationship which should ultimately be developed from the differential form (14).

THE STREAM FUNCTION, VELOCITY AND RELATIVE ENTHALPY IN THE FRAME OF THE BLADE

Since ideal flow has been assumed in the moving frame of the blade, the stream function, ψ, is the usual function modified for motion along the y axis.

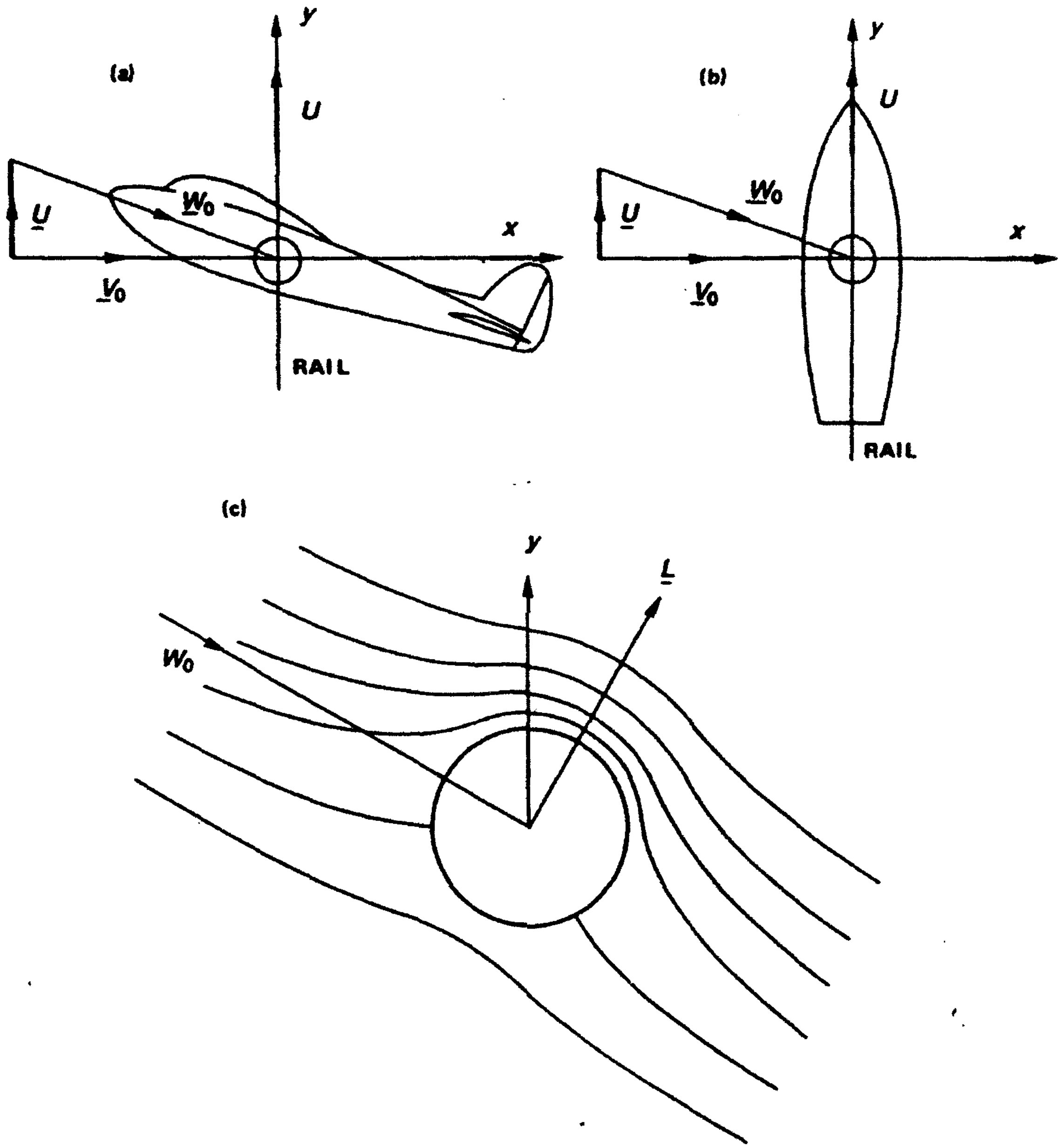

Figure 2. The single-blade linear turbine: (a) sailplane version; (b) sailboat version, and (c) the rotating cylinder blade.

$$\psi = -W_{oy}(1 - a^2/r^2)r\cos\theta + W_{ox}(1 - a^2/r^2)r\sin\theta$$

$$+ (\Gamma/2\pi)\ln(r/a)\ . \tag{39}$$

The constant a is the radius of the cylinder. The cartesian velocity components are obtained by the usual transformation [10] as follows:

$$W_x = W_{ox} + \frac{a^2W_{ox}(y^2 - x^2)}{(x^2+y^2)^2} - \frac{2a^2W_{oy}xy}{(x^2+y^2)^2}$$

$$+ \frac{\Gamma}{2\pi}\ \frac{y}{(x^2+y^2)}\ , \tag{40}$$

$$W_y = W_{oy} - \frac{2a^2W_{ox}xy}{(x^2+y^2)^2} + \frac{a^2W_{oy}(x^2 - y^2)}{(x^2+y^2)^2}$$

$$- \frac{\Gamma}{2\pi}\ \frac{x}{(x^2+y^2)}\ . \tag{41}$$

Now the relative vorticity must vanish because potential flow cannot have vorticity. A check of the vorticity in the relative frame shows that indeed it vanishes. Also, the time-dependent term vanishes.

THE SUBSTANTIAL TOTAL ENTHALPY DERIVATIVE

In the linear two-dimensional system, the differential form of the turbomachinery equation (14) is simplified because the rotation vanishes.

$$\frac{Dh_o}{Dt} = \bar{U}\cdot(\bar{W}\cdot\nabla)\bar{W}\ . \tag{42}$$

Since the vorticity vanishes

$$\bar{W}\cdot\nabla\bar{W} = \nabla W^2/2\ , \tag{43}$$

and

$$\frac{Dh_o}{Dt} = \bar{U}\cdot\nabla W^2/2 = \frac{U}{2}\frac{\partial W^2}{\partial y_w}\ . \tag{44}$$

The integrated substantial total enthalpy rate per unit length [where the subscript on y in (44) has been dropped] is

$$\frac{DH_o'}{Dt} = \int\int \frac{\varrho U}{2}\ \frac{\partial(W_x^2 + W_y^2)}{\partial y}dydx\ . \tag{45}$$

Integration of (45) will be performed over all space per unit length z of the blade. The choice of time is immaterial since the fluid dynamics are steady state in the moving frame and the thermodynamic rates over all space are invariant with time. It is understood that the integration applies only to the fluid domain and that boundaries at solid walls are observed.

Now without going into details [10] the integral of the total derivative in (45) is

$$\frac{DH_o'}{Dt} = -\varrho U \int_{-a}^{a} \left(\frac{4UW_{ox}\, x(a^2-x^2)^{1/2}(2x^2-a^2)}{a^4} \right.$$

$$+ \frac{2W_{ox}\Gamma x(a^2-x^2)^{1/2}}{\pi a^4} + \frac{8W_{ox}Ux(a^2-x^2)^{1/2}}{a^2}$$

$$+ \frac{W_{ox}\Gamma(a^2-x^2)^{1/2}}{\pi a^4} + \frac{4W_{ox}Ux(a^2-x^2)^{1/2}(a^2-2x^2)}{a^4}$$

$$\left. + \frac{W_{ox}\Gamma(a^2-x^2)^{1/2}(a^2-2x^2)}{\pi a^4} \right) dx \ . \qquad (46)$$

Note that only odd terms in y make any contribution to (46). Since the first and fifth terms cancel, only four terms remain. The integration with respect to x is performed through a transformation employing

$$x = a\cos\theta \ , \qquad (47)$$

with integration limits given by

$$\theta = \pi \ \ when \ \ x = -a \ ,$$

$$\theta = 0 \ \ when \ \ x = a \ . \qquad (48)$$

Making the substitutions

$$\frac{DH_o'}{Dt} = -\varrho U \left(-\frac{2W_{ox}\Gamma}{\pi} \int_{\pi}^{0} \cos^2\theta \sin^2\theta d\theta \right.$$

$$-8W_{ox}Ua \int_{\pi}^{0} \cos\theta \sin^2\theta d\theta - \frac{W_{ox}\Gamma}{\pi} \int_{\pi}^{0} \sin^2\theta d\theta$$

$$\left. -\frac{W_{ox}\Gamma}{\pi} \int_{\pi}^{0} \sin^2\theta(1-2\cos^2\theta) d\theta \right) . \qquad (49)$$

In (50) the second integral makes no contribution because it is antisymmetric. The first integral cancels the second term in the last integral to yield from the surviving terms

$$\frac{DH_o'}{Dt} = -\varrho W_{ox}\Gamma U = -\varrho V_{ox}\Gamma U \ . \tag{50}$$

Equation (50) is identical with (38) and this result illustrates a useful application of the differential form and constitutes confirmation of the validity of the differential turbo-machinery Equation (14). For the linear case, the energy transfer rate of the rotor is proportional to the component of the kinetic energy gradient parallel to the moving rotor.

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A differential equation describing the energy transfer between a fluid and a body moving in that fluid was derived. The derivation, based upon the Coriolis form of the Navier-Stokes Equation, contains a rigorous viscous correction. For inviscid ideal cases, the equation demonstrates that the rate of total enthalpy transfer from (or to) the system is a function of the transverse component of the pressure gradient. Therefore, for practical turbomachinery rotors, the derivative, $\partial p/\partial\theta$ can never vanish.

On integration of the differential equations, a form of the Euler Turbomachinery Equation with viscous correction is derived. The resultant form contains two distinct work rate terms for the axial and radial components of the flow. The fact that integration yields a result which approximates the classic Euler Turbomachinery Equation constitutes confirmation of the derivation.

An application of the equation to an ideal infinite linear cylinder with bound vorticity was developed, yielding the expected known result.

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